SOLUTION OF SOME PROBLEMS ON THE ONE-DIMENSIONAL UNSTEADY MOTION OF A CONDUCTING GAS UNDER THE EFFECT OF STRONG ELECTROMAGNETIC FIELDS

(RESHENIE NEKOTORYKH ZADACH OB ODNOMERNOM NEUSTANOVIVSHEMSIA DVIZHENII PROVODIASHCHEGO GAZA POD DEISTVIEM SIL'NYKH ELEKTROMAGNITNYKH POLEI)

PMM Vol.28, № 4, 1964, pp.664-669

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(Received April 6, 1964)

The system of equations of one-dimensional motion of a conducting gas (plasma) under the effect of strong electromagnetic fields for the case of small magnetic Reynolds numbers is reduced to one equation by means of an appropriate selection of variables. Self-similar solutions of this equation are investigated and a solution is also given for the problem of the motion of a conducting gas in an unlimited channel under the effect of an alternating electromagnetic field.

1. Let us consider the one-dimensional unsteady motion of a plasma (conducting gas). In addition to the customary assumptions for which the motion of a plasma may be considered one-dimensional, let us also assume that in the case under consideration the electromagnetic forces are much greater than the pressure forces so that the term containing the pressure gradient in the equation of motion may be neglected. There are appropriate estimates for when this assumption is valid (e.g. in [1]). Hence, the equations of one-dimensional unsteady motion of a plasma under the effect of electromagnetic forces are

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \qquad (1.1)$$

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} = -\frac{1}{4\pi}H\frac{\partial H}{\partial x}, \qquad \frac{\partial H}{\partial x} = -\frac{4\pi\sigma}{c}\left(E - \frac{1}{c}uH\right) \qquad (1.2)$$

Here t is the time, x the coordinate along the channel axis, ρ the density, u the plasma velocity, H the magnetic and E the electric field intensities, σ the conductivity which will be considered constant and c the velocity of light in vacuo.

In place of x let us introduce the new independent variable m by means of Formula

$$m = \int_0^x \rho\, dx - \int_0^t \rho_{00} u_{00}\, dt \qquad (1.3)$$

where ρ_{00} and u_{00} are the values of the density ρ and the velocity u at $x = 0$. The physical meaning of the Lagrange coordinate m may be clarified by noting that the first integral in (1.3) is the mass of gas referred to unit cross-sectional area of the channel which is between the initial section and the section with the coordinate x at a given time t . The second integral in (1.3) yields the whole mass of gas which has flowed into the channel through the $x = 0$ section within the time from $t = 0$ to the considered time t . Therefore, m is the negative of the mass of gas included between the particles in the section of the channel with coordinate x at time t and the gas particles which passed the origin at time $t = 0$.

Differentiating (1.3) with respect to x and t we obtain

$$\frac{\partial m}{\partial x} = \rho, \qquad \frac{\partial m}{\partial t} = \int_0^x \frac{\partial \rho}{\partial t}\, dx - \rho_{00} u_{00} = -\rho u$$

Here the continuity equation (1.1) has been used for the last relation with the result that

$$\int_0^x \frac{\partial \rho}{\partial t}\, dx = -\rho u + \rho_{00} u_{00}$$

Remarking also that

$$\frac{\partial x}{\partial m} = \frac{1}{\partial m / \partial x} = \frac{1}{\rho}, \qquad \frac{\partial x}{\partial t} = -\frac{\partial m}{\partial t}\frac{\partial x}{\partial m} = u \qquad (1.4)$$

we transform (1.1), (1.2) to the independent t and m variables

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial u}{\partial m} = 0, \quad \frac{\partial u}{\partial t} = -\frac{1}{4\pi} H \frac{\partial H}{\partial m}, \quad \rho \frac{\partial H}{\partial m} = -\frac{4\pi\sigma}{c}\left(E - \frac{1}{c} uH\right) \qquad (1.5)$$

This system of equations may easily be reduced to one equation. Let us execute this transformation in the case when the magnetic Reynolds number is such that the induced magnetic field may be neglected in comparison with the external magnetic field.

Eliminating $\partial H/\partial m$ in this case, we obtain in place of (1.5)

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial u}{\partial m} = 0, \qquad \frac{\partial u}{\partial t} = \frac{\sigma}{c\rho} H\left(E - \frac{1}{c}\, uH\right) \qquad (1.6)$$

where E and H are given functions of x and t

Let us now introduce a new desired function of x . Using (1.4) we obtain that the first equation of (1.6) will be satisfied identically and the second will yield

$$\frac{\partial^2 x}{\partial t^2} = \frac{\sigma H^2}{c^2}\left(\frac{cE}{H} - \frac{\partial x}{\partial t}\right)\frac{\partial x}{\partial m} \qquad (1.7)$$

If the velocity distribution u_0 and the density ρ_0 of the plasma along the length of the channel are given at the initial instant, then $m(x_0)$ can be determined from (1.3). Evaluating the inverse function $x_0(m)$ and noting that $u = x_t'$, we obtain the initial conditions in the form

$$x = x_0(m), \qquad x_t' = u_0(m) \quad \text{for } t = 0 \tag{1.8}$$

In the general case of a bounded channel, it is necessary to assign boundary conditions in addition to the initial conditions (1.8). Assuming that the entrance to the channel is at the $x = 0$ section, it is possible to give the velocity and discharge (density) of the gas as a function of the time as the boundary conditions at $x = 0$.

Assignment of these quantities permits the determination of $m = m(t)$ at $x = 0$.

Hence, the boundary conditions in the m and t variables will be

$$x = 0, \qquad x_t' = u_{00}(t) \quad \text{for } m = m(t)$$

Equation (1.7) is a nonlinear second order equation in the unknown function x. It is very difficult to investigate its solution in the general form.

Hence, let us first consider in detail that particular case when the motion is such that $E \gg c^{-1} u H$ and the external electromagnetic field depends only on time. Then, the second term on the right-hand side of (1.7) may be discarded as compared to the first. We hence obtain

$$\partial^2 x / \partial t^2 = f(t) \partial x / \partial m \tag{1.9}$$

where for brevity we have introduced the notation

$$(\sigma / c) E(t) H(t) = f(t)$$

2. Let us consider self-similar motions first. Let the n dimensional constants $a_1, \ldots, a_n$ enter into (1.9), the boundary and initial conditions. Let us take the quantities x, t and m as independent dimensions. Without limiting the generality, we may assume that a_1 contains the dimension x. Then combining $a_2, \ldots, a_n$ with a_1 it is possible to obtain $n-1$ new constants $a_2', \ldots, a_n'$ instead of the $a_2, \ldots, a_n$, where these former will not contain the dimension x. In order that (1.9) should admit self-similar solutions, it is necessary that among the constants $a_2', \ldots, a_n'$ there be at least one with an independent dimension [2]. The function $f(t)$ in (1.9) may be represented as

$$f(t) = a_0 f_1(t / t_0)$$

where the dimension of the constant is $[a_0] = [m/t^2]$ and f_1 is a nondimensional function. Since the dimensions of a_0 and t_0 are independent, the self-similar solutions will exist only when $f(t)$ has the form

$$f(t) = kt^\alpha \tag{2.1}$$

Furthermore, without carrying out computations, let us note that if the functions in the initial or boundary conditions are not identically zero, they sould have a definite form so that no new constants with independent dimension would appear and the solution would still be self-similar. Namely, the initial and boundary conditions should be

$$\begin{aligned} x &= a_1 m^{\frac{\beta+1}{2+\alpha}}, \quad x_t' = a_2 m^{\frac{\beta}{2+\alpha}} \quad &\text{for } t = 0 \\ x &= 0, \quad x_t' = at^{\beta} \quad &\text{for } m = a_3 t^{2+\alpha} \end{aligned} \tag{2.2}$$

Hence (1.9) will have a self-similar solution of the form

$$x = at^{\beta+1}\varphi(\lambda) \tag{2.3}$$

where φ is a nondimensional function and the nondimensional variable λ equals

$$\lambda = \frac{1}{k} mt^{-(2+\alpha)} \tag{2.4}$$

Substituting (2.3) into (1.9) we obtain an ordinary differential equation for the determination of the function $\varphi(\lambda)$

$$\left(\frac{2+\alpha}{k}\right)^2 \lambda^2\varphi'' - \left[1 + \frac{2+\alpha}{k}\left(2\beta + 1 - \frac{2+\alpha}{k}\right)\lambda\right]\varphi' + \beta(\beta+1)\varphi = 0 \tag{2.5}$$

The solution of (2.5) contains two arbitrary constants. Hence, it is impossible to satisfy two boundary and two initial conditions at the same time if the constants therein are given arbitrarily (i.e. the motion will not be self-similar). The question is simplified in two cases. Firstly, if the motion occurs in an unbounded channel. This problem will be considered below since a solution even for non-self-similar motion has successfully been obtained for the unbounded channel. Secondly, when there is no plasma in the channel at the initial instant, i.e. the initial parameters are zero. A number of problems of practical interest concerning the initial period of operation of a plasma-acceleration channel when the channel starts to be filled with plasma and the electric and magnetic fields grow from zero after the system has been connected, reduces to this case.

Reducing the boundary conditions (2.2) to nondimensional form, we obtain two boundary conditions for φ

$$\varphi = 0, \quad \varphi' = -\frac{k}{(2+\alpha)\lambda_0} \quad \text{for } \lambda = \frac{a_3}{k} = \lambda_0 \tag{2.6}$$

As an example, let us find the final solution of the following problem: plasma starts to flow into a channel at time $t = 0$ with the velocity $u = \alpha t^2$ and the constant density ρ. The electric and magnetic fields grow such that $f(t) = 1.5t$.

From (1.3), (2.1), (2.2), (2.4) and (2.6) we find that in the example considered we will have

$$\alpha = 1, \quad \beta = 2, \quad k = 1.5, \quad \lambda_0 = -{}^1/_3 a\rho$$

Equation (2.5) and boundary conditions (2.6) take the form

$$4\lambda^2\varphi'' - (1 + 6\lambda)\varphi' + 6\varphi = 0 \qquad (2.7)$$

$$\varphi = 0, \qquad \varphi' = -1/(2\lambda_0) \quad \text{for } \lambda = \lambda_0 \qquad (2.8)$$

The general integral of (2.7) will be

$$\varphi = (1 + 6\lambda)\left[C_1 \int_{\lambda_0}^{\lambda} \lambda^{1.5} \frac{e^{-1/(4\lambda)} d\lambda}{(1 + 6\lambda)^2} + C_2\right] \qquad (2.9)$$

where C_1 and C_2 are arbitrary constants.

Determining their values from the boundary conditions (2.8), we obtain

$$C_1 = -0.5\,(1 + 6\lambda_0)\,\lambda_0^{-2.5} e^{1/(4\lambda_0)}, \qquad C_2 = 0 \qquad (2.10)$$

Substituting (2.9) and (2.10) into (2.3), we find the desired expression for x

$$x = aC_1 t^3\,(1 + 6\lambda) \int_{\lambda_0}^{\lambda} \frac{\lambda^{1.5}\, e^{-1/(4\lambda)}}{(1 + 6\lambda)^2}\, d\lambda$$

3. Let us now carry out the solution of the problem of plasma motion in an unbounded channel in the general case. Let the velocity distribution and the density along the channel length be known at the initial instant in a channel unbounded at both ends. It is required to determine the motion originating from this initial state under the effect of a given constant electromagnetic field along the length of the channel but which changes with time.

As has been shown above, the question reduces to seeking the solution of (1.9) which will satisfy the initial conditions (1.8). Let us seek the solution in the form of the infinite series

$$x = \sum_{n=0}^{\infty} [x_0^{(n)}(m)\,\psi_n(t) + u_0^{(n)}(m)\,\chi_n(t)] \qquad (3.1)$$

where $x_0^{(n)}$ and $u_0^{(n)}$ are the nth derivative of x_0 and u_0 .

Substituting (3.1) into (1.9), we find that for (3.1) to be a solution it is necessary that

$$\psi_n = \int_0^t dt \int_0^t f(t)\,\psi_{n-1}\,dt, \qquad \chi_n = \int_0^t dt \int_0^t f(t)\,\chi_{n-1}\,dt \qquad (3.2)$$

and we find from the initial conditions (1.8)

$$\psi_0 = 1, \qquad \chi_0 = t \qquad (3.3)$$

Formulas (3.1) to (3.3) yield the solution of the formulated problem in the general case. Let us illustrate their use in a specific example.

At $t = 0$ let the plasma velocity be $u \equiv 0$ and let the density be distributed according to the law

$$\rho = 1 / 2\sqrt{kx} \quad \text{for } x > 0, \qquad \rho \equiv 0 \quad \text{for } x < 0$$

It is required to determine what motion will originate from this initial state under the effect of a periodic electromagnetic field, i.e. for $f = A\sin\omega t$.

We calculate the connection between x and m at $t = 0$ by means of (1.3). Performing the computations we obtain the initial conditions in the considered example in the form

$$x = km^2, \qquad x_t' = 0 \quad \text{for } t = 0 \tag{3.4}$$

By means of (3.2) we calculate

$$\psi_1 = A\omega^{-2}(\omega t - \sin\omega t) \tag{3.5}$$

$$\psi_2 = {}^1/_8 A^2\omega^{-4}(17 - 2\omega^2t^2 - 8\omega t\sin\omega t - 16\cos\omega t - \cos 2\omega t)$$

It is necessary to evaluate ψ_n for $n \geqslant 3$ and χ_n since for the given boundary conditions (3.6)

$$x_0' = 2km, \quad x_0'' = 2k, \quad x_0^{(n)} \equiv 0 \quad \text{for } n \geqslant 3, \qquad u_0^{(n)} \equiv 0 \quad \text{for } n \geqslant 0$$

Substituting (3.4), (3.5) and (3.6) into (3.1), we obtain the final formulas describing the plasma motion

$$x = k\,[m^2 + 2A\omega^{-2}(\omega t - \sin\omega t)\,m + {}^1/_4 A^2\omega^{-4}(17 - {} - 2\omega^2t^2 - 8\omega t\sin\omega t - 16\cos\omega t - \cos 2\omega t)] \tag{3.7}$$

Differentiating (3.7) with respect to time, we calculate the particle velocity

$$u = k[2A\omega^{-1}(1 - \cos\omega t)\,m - {} - A^2\omega^{-3}(\omega t - 2\sin\omega t + 2\,\omega t\cos\omega t - {}^1/_2\sin 2\omega t)] \tag{3.8}$$

Eliminating m from (3.7) and (3.8), we may find the dependence of the particle velocity on the coordinate x. In order not to write down the awkward and poorly illustrative formulas, let us perform these computations only for the times $t = (2n+1)\pi/\omega$ and $t = 2n\pi/\omega$. Omitting the intermediate formulas we arrive at the final result

$$u = \frac{Ak}{\omega}\left\{4\left(\frac{1}{k}\,x + \frac{A^2}{2\omega^4}\,[3\,(2n+1)^2\pi^2 - 16]\right)^{1/2} - \frac{3A}{\omega^2}(2n+1)\,\pi\right\} \quad \text{for } t = \frac{(2n+1)\pi}{\omega} \tag{3.9}$$

$$u = -\,6kA^2n\pi\omega^{-3} \qquad \text{for } t = 2n\pi\omega^{-1} \tag{3.10}$$

It is seen from (3.9) and (3.10) that the plasma will perform oscillatory motion in the sense that the particle velocity periodically changes sign. However, these will not be oscillations of the particles around some fixed equilibrium position. In fact, it follows from (3.7) and the initial conditions (3.4) that a particle at the point x_0 in the right half of the channel at the initial instant would, after n periods, i.e. at $t = 2n\pi/\omega$ have deviated from its original position by

$$x - x_0 = 2Akn\pi(2\omega^2 m - An\pi) / \omega^4 \qquad (3.11)$$

It is seen from (3.11) that for a fixed particle, i.e. for a fixed m, sooner or later the $x - x_0$ would become negative as n increases. Hence, for particles for which $m > A\pi/2\omega^2$, first $x - x_0$ grows and then it decreases and becomes negative. Hence, the plasma performs a complex motion in which the plasma particles move, while oscillating on the average first to the right and then to the left. A peculiar wave seems to be propagated along the plasma particles, which first slows down the average motion of the particles to the right and then involving them in average motion to the left.

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Translated by M.D.F.